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A MINIMUM PRINCIPLE FOR THE SMALLEST EIGENVALUE FOR SECOND ORDE--ETC(U)

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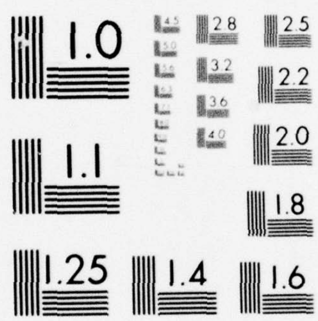


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| REPORT DOCUMENTATION PAGE | | READ INSTRUCTIONS BEFORE COMPLETING FORM |
|---|---|---|
| 1. REPORT NUMBER: AFOSR-TR-78-0121 | 2. GOVT ACCESSION NO. | 3. RECIPIENT'S CATALOG NUMBER |
| 4. TITLE (and Subtitle): A MINIMUM PRINCIPLE FOR THE SMALLEST EIGENVALUE FOR SECOND ORDER LINEAR ELLIPTIC EQUATIONS WITH NATURAL BOUNDARY CONDITIONS | | 5. TYPE OF REPORT & PERIOD COVERED: Interim rept. |
| | | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s): Charles J. Holland | | 8. CONTRACT OR GRANT NUMBER(s): AFOSR-77-3286 |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS: Purdue University, Department of Mathematics, West Lafayette, IN 47907 | | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F 2304/A1 |
| 11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Office of Scientific Research/NM, Bolling AFB, DC 20332 | | 12. REPORT DATE: August 1977 |
| | | 13. NUMBER OF PAGES: 18 (22p.) |
| 14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) | | 15. SECURITY CLASS. (of this report): UNCLASSIFIED |
| | | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)



18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper we give a new characterization of the smalles eigenvalue for second order linear elliptic partial differential equations, not necessarily self-adjoint, with both natural and Dirichlet boundary conditions, and also give a new alternative numerical method for calculating both the smalles eigenvalue and corresponding eigenvector in the case of natural boundary conditions. The smalles eigenvalue, if appropriate sign changes are made, determines the stability of equilibrium solutions to certain second order nonlinear partial differential equations. The corresponding eigenvector enables one to determine the first approximation of the solution of the nonlinear equation to variations of the

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20. Abstract

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AFOSR-TR- 78 - 0121

# A Minimum Principle for the Smallest Eigenvalue for Second Order Linear Elliptic Equations with Natural Boundary Conditions

Charles J. Holland
Purdue University

August 1978

## Accompanying Statement

In this paper we give a new characterization of the smallest eigenvalue for second order linear elliptic partial differential equations, not necessarily self-adjoint, with both natural and Dirichlet boundary conditions, and also give a new alternative numerical method for calculating both the smallest eigenvalue and corresponding eigenvector in the case of natural boundary conditions. The smallest eigenvalue, if appropriate sign changes are made, determines the stability of equilibrium solutions to certain second order nonlinear partial differential equations. The corresponding eigenvector enables one to determine the first approximation of the solution of the nonlinear equation to variations of the initial condition from the equilibrium solution. These nonlinear equations are important in the applications. For these reasons it is important to have these characterizations of the smallest eigenvalue and eigenvector.

Our method converts the determination of the eigenvalue and eigenvector to determining the solution of a stationary

stochastic control problem. This latter problem is solved and from it a numerical scheme arises naturally. This method appears to have applications in solving other problems.

This research was supported in part by AFOSR Grant 77-3286.

A Minimum Principle for the Principal Eigenvalue
for Second Order Linear Elliptic Equations
With Natural Boundary Conditions*

Charles J. Holland
Courant Institute of Mathematical Sciences
and
Purdue University

## 1. Introduction

In this paper we give a new characterization of the principal eigenvalue $\lambda^*$ for the eigenvalue equation

$$(1) \qquad -\nabla \circ (u_x a) + u_x b + cu = \lambda \rho u \quad \text{in } \Omega$$

with the natural boundary condition $u_x an = 0$ on $\partial\Omega$. Above a is a positive definite matrix for each $x \in \bar{\Omega}$, b is a vector, and the functions $a_{ij}$, $b_i$, c, $\rho$ are functions of class $C^2(\bar{\Omega})$ with $\rho > 0$ in $\bar{\Omega}$. In Theorem 1 we establish that $\lambda^*$ satisfies

$$(2) \quad \lambda^* = \min_{\phi\in\Phi} \max_{V\in C^1(\Omega)} \left\{ \int_\Omega \phi_x a \phi_x^T + c\phi^2 + b^T(4a)^{-1}b\phi^2 + \phi\phi_x b dx - \frac{1}{2}\int_\Omega [(2a)^{-1}b - V_x^T]^T(2a)[(2a)^{-1}b - V_x^T]\phi^2 dx \right\}$$

where $\Phi$ is the set of functions $\phi$ of class $C^2(\Omega) \cap C(\bar{\Omega})$ with $\phi^2 > 0$ in $\bar{\Omega}$ and $\int_\Omega \rho\phi^2 = 1$. If $b = 0$ the operator defined by (1) is self-adjoint and (2) reduces to Rayleigh-Ritz. If not, then the expression (2) can be considered as a generalized Rayleigh-Ritz expression.

Recently, using different methods, Donsker-Varadhan [1] have derived a similar expression in the case of zero Dirichlet boundary conditions and $\rho = 1$. In Section 3 we outline our approach to the Dirichlet case.

Finally, in Section 4 we made some remarks concerning a possible numerical scheme for computing approximations to $\lambda^*$ and the corresponding normalized eigenfunction $u^*$. This scheme does not use (2), but instead involves the solution of a set of nonlinear difference equations. These equations result from a discretization of equation (5) below. The difference equations are then related to a stationary stochastic control problem.

## 2. Natural boundary conditions

THEOREM 1. *Assume, in addition to the above, that the functions* $a_{ij}$, $b_i$, $c$ *are of class* $C^2(\bar{\Omega})$ *and that* $\Omega$ *is a domain of class* $C^2$. *Then* (2) *is true*.

Proof: Equation (1) can be written in the form

$$(-3)\qquad -\operatorname{trace} a u_{xx} + u_x(b+d) + cu = \lambda\rho u$$

for an appropriate vector function d. As a consequence of the Krein-Rutman theory of positive operators and the maximum principle, we let $u^*$ denote an eigenfunction corresponding to the principal eigenvalue $\lambda^*$ such that $u^* > 0$ in $\bar{\Omega}$ and $\int_\Omega \rho(x)u^*(x)^2 dx = 1$. We now make the change of variables $u^* = \exp(-\psi)$ obtaining

+ $$(4)\qquad \operatorname{trace} a\psi_{xx} - \psi_x a \psi_x^T - \psi_x(b+d) + c = \lambda^*\rho\ .$$

We have adopted the convention that all vectors are column vectors except gradients which are row vectors. We now rewrite (4) in the form

$$(5)\qquad \operatorname{tr} a\psi_{xx} + \min_w\, [\psi_x w + (b+d+w)^T(4a)^{-1}(b+d+w) + c] = \lambda^*\rho$$

where the min in (5) is for each x taken over vectors $w \in R^n$. Note this minimum is obtained for

$$(6)\qquad w = -b-d -2(\psi_x a)^T$$

or

$$(7)\qquad w = -b-d + 2(u^*)^{-1}(u^*_x a)^T\ .$$

The boundary condition for $\psi$ become $\psi_x a n = 0$ on $\partial\Omega$.

Now let $\mathcal{W}$ denote the set of functions w of the form $w = -b-d+2(u)^{-1}(u_x a)^T$ for some smooth positive function u in $\bar{\Omega}$ such that $\int_\Omega \Omega(x)u^2(x)dx = 1$. For any $w \in \mathcal{W}$ we have the inequality

$$(8) \qquad \operatorname{tr} a\psi_{xx} + \psi_x w + (b+d+w)^T(4a)^{-1}(b+d+w) + c \geq \lambda^* \rho \ .$$

Define for each $w \in \mathcal{W}$, the operator L by

$$(9) \qquad Lp = \nabla \circ (p_x a) - \nabla \circ ((d+w)p) = 0 \ .$$

Then the equation $Lp = 0$ in $\Omega$ with the boundary condition

$$(10) \qquad [p_x a - p(d+w)^T] \cdot n = 0 \quad \text{on } \partial\Omega$$

has a smooth solution $p_w$ which is positive in $\bar{\Omega}$ and satisfies the normalizing condition $\int_\Omega \rho(x)p_w(x)dx = 1$. The existence of the positive solution $p_w$ follows from the Krein-Rutman theory and the maximum principle. The Krein-Rutman theory gives that the eigenvalue equation $Lp = \lambda p$ with boundary condition (10) has a positive solution in $\Omega$ when $\lambda$ is the principal eigenvalue. Integrating the equality $Lp = \lambda p$ over the domain $\Omega$, one obtains that $\lambda = 0$ and hence $Lp = 0$ has a positive solution in $\Omega$. Positivity on the boundary follows from (10) with use of maximum principle.

For each $w \in \mathcal{W}$ we multiply (8) by $p_w$ and integrate over $\Omega$ obtaining

$$(11) \qquad \int_\Omega [(b+d+w)^T(4a)^{-1}(b+d+w) + c]p_w dx \geq \lambda^* \ .$$

Equality in (11) holds only when w is given by (7). Thus we have

$$(12) \qquad \min_{w\in\mathcal{W}} \int_\Omega [(b+d+w)^T(4a)^{-1}(b+d+w) + c]p_w dx = \lambda^* .$$

The proof of the theorem consists in reformulating (12) by discovering the relationship between w and $p_w$.

We break the proof into three cases.

Case 1. b = 0 (the self-adjoint case). For any $w = (-d+2(u)^{-1}(u_x a)^T) \in \mathcal{W}$, $p_w$ is given by $u^2$. Simply check that $u^2$ satisfies $L(u^2) = 0$ and the boundary condition (10). Substituting w and $p_w$ into (12), one obtains that (12) can be rewritten as

$$(13) \qquad \lambda^* = \min_{u\in\Phi} \left[\int_\Omega u_x a u_x^T + cu^2 dx\right] .$$

Equation (13) is also (2), and (13) is the standard Rayleigh-Ritz expression.

Case 2. $(2a)^{-1}b$ is a gradient. Let k be such that $k_x = (2a)^{-1}b$. (This is the case when the original equation could have been made self-adjoint by multiplying the equation by $e^q$ for an appropriate function q.) Now any control $w \in \mathcal{W}$ of the form $w = -b-d+2\phi^{-1}(\phi_x a)^T$ can be rewritten as $\tilde{w} \in \mathcal{W}$ where

$$(14) \qquad \tilde{w} = -d+2u^{-1}(u_x a)^T$$

with $u = c\phi e^k$ for an appropriate constant c. Thus we need only use "controls" of the form (14). Recall that $p_{\tilde{w}} = u^2$. Substituting into (12) we obtain

$$(15) \qquad \lambda^* = \min_{u\in\Phi} \int_\Omega [u_x a u_x^T + cu^2 + b^T(4a)^{-1} b u^2 + u u_x b]dx$$

which is (2) for this case.

Case 3. ($(2a)^{-1}b$ is arbitrary). We wish to write each control

$$(16) \qquad w = -b - d + 2\phi^{-1}(\phi_x a)^T \in \mathcal{W}$$

in the form

$$(17) \qquad w = -d + 2u^{-1}(u_x a)^T + z$$

for smooth functions u, z such that $u > 0$ in $\bar{\Omega}$, $\int_\Omega \rho(x)u^2(x)dx = 1$, and $p_w = u^2$ where $p_w$ is the solution to $Lp_w = 0$ with boundary condition (10). Moreover we require that $z = -b + 2(W_x a)^T$ for some function W. Let us show that this is possible.

First, for any $\phi$ given in (16) we must have that $u = \phi e^{-W}$. From the equation $Lp = 0$ with boundary condition (10) we see that u, z must satisfy

$$(18) \qquad \begin{aligned} \nabla \circ (u^2 z) &= 0 \quad \text{in } \Omega \\ z \cdot n &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

if $p_w$ for w given by (17) is to be $p_w = u^2$. Using the condition $u = \phi e^{-W}$ we obtain the equation

$$(19) \qquad \nabla \circ (\phi^2 e^{-2W}(-b + 2(W_x a)^T) = 0 \quad \text{in } \Omega$$

with boundary condition

$$(20) \qquad (-b^T + 2(W_x a)) \cdot n = 0 \quad \text{on } \partial\Omega$$

which must be solved for W for each $\phi$. The equations (19), (20) have a solution provided the equations

$$
(21) \qquad \begin{aligned} &\nabla \circ (\phi^2 Gb + \phi^2 (G_x a)^T) = 0 \quad \text{in } \Omega , \\ &(-\phi^2 G b^T + \phi^2 G_x a)\cdot n = 0 \quad \text{on } \partial\Omega \end{aligned}
$$

have a positive solution G. The solution of (19), (20) is obtained from (21) through the substitution $G = e^{-2W}$. Now (21) has a positive solution by a repetition of the earlier argument concerning the existence of a positive solution to $Lp = 0$ with boundary condition (10).

Thus every control of the form (16) can be written in the form

$$
(22) \qquad w = -d + 2u^{-1}(u_x a)^T + (-b + 2(W_x a)^T)
$$

with $p_w = u^2$ and u, W satisfy

$$
(23) \qquad \begin{aligned} &\nabla \circ (u^2(-b + 2(W_x a)^T)) = 0 \quad \text{in } \Omega , \\ &(-b^T + 2(W_x a))\cdot n = 0 \quad \text{on } \partial\Omega . \end{aligned}
$$

Before evaluating the cost of using controls of the form (22), (23) we need to observe the following. For any smooth $u > 0$ in $\overline{\Omega}$ consider the problem of minimizing

$$
(24) \qquad I(V) = \int_\Omega ((2a)^{-1}b - V_x^T)^T (2a)((2a)^{-1}b - V_x^T) u^2 dx
$$

over the class of smooth functions V. Then W given by (23) minimizes (24) and moreover

$$(25) \qquad \int_\Omega Q_x(-b + 2(W_x a)^T)u^2 dx = 0$$

for any smooth function Q. The proof of (24), (25) is a simple exercise in the calculus of variations. We will use (25) below when $Q = \ell n\ u$ and $Q = W$.

Let us now use a control of the form (22). Then the left hand side of (11) can be written as the sum of four terms

$$(26) \qquad \int_\Omega [u_x a u_x^T + cu^2 + b^T(4a)^{-1}bu^2 + uu_x b]dx$$

$$+ \int_\Omega [(-2a)^{-1}b + W_x^T]^T a[(-2a)^{-1}b + W_x^T]u^2 dx$$

$$+ \int_\Omega (\nabla\ \ell n\ u)(-b + 2(W_x a)^T)u^2 dx$$

$$+ \int_\Omega b^T(2a)^{-1}(-b + 2(W_x a)^T)u^2 dx\ .$$

The third term is zero by virtue of (25). Using (25) we can add

$$0 = \int -W_x(-b + 2(W_x a)^T)u^2 dx$$

to the fourth term to obtain -2 times the second integral in (26). Therefore the sum of the second, third and fourth integrals can be written as

$$(27) \qquad -\frac{1}{2}\int_\Omega [(2a)^{-1}b - W_x^T]^T(2a)[(2a)^{-1}b - W_x^T]u^2 dx\ .$$

Using the definition of the minimum problem for I(V) and the fact that W minimizes I(V), one can rewrite (27) as

$$(28) \qquad \max_{V \in C^1(\bar{\Omega})} \; -\frac{1}{2}\int_{\Omega} [(2a)^{-1}b - V_x^T]^T(2a)[(2a)^{-1}b - V_x^T]u^2 dx \; .$$

Substituting this into (26) completes the proof of the theorem.

Remarks. Although we have given a proof that avoids the explicit use of probabilistic ideas, we were motivated by probabilistic concerns. Take $\rho = 1$, then (5) with the boundary condition $\psi_x an = 0$ is the "verification equation" for a stationary stochastic control problem. With each Lipschitz function w associate the Ito stochastic differential equation

$$(29) \qquad d\xi = w(\xi)dt + \sqrt{2}\,\sigma(\xi)db(t) \; , \qquad \sigma\sigma^T = a \; ,$$

b = brownian motion, with reflection in the interior of $\Omega$ in the direction -an where n is the outer normal. Then $p_w$ given by (9), (10) is the invariant density of the process.. Associate with each "control" w the cost

$$(30) \qquad C(w) = \int_{\Omega} [(b+d+w)^T(4a)^{-1}(b+d+w) + c]p_w dx \; .$$

Using equation (5) and the Ito stochastic differential rule one obtains that the minimum of C(w) over Lipschitz control function is given by $\lambda^*$. Using the approach one can derive the alternative representation (recall $\rho = 1$)

$$(31) \quad \lambda^* = \min_{\phi \in \Phi} \; \min_{\substack{z \in C^1(\Omega) \cap C(\bar{\Omega}) \\ \nabla \circ (\phi^2 z) = 0 \text{ in } \Omega \\ z \cdot n = 0 \text{ on } \partial\Omega}} \Big\{ \int [\phi_x a \phi_x^T + c\phi^2 + \phi\phi_x b + b^T(4a)^{-1}b\phi^2]dx$$

$$+ \int_{\Omega} [(z^T(4a)^{-1}z + z^T(2a)^{-1}b)\phi^2 + \phi\phi_x z]dx \Big\} \; .$$

## 3. Dirichlet boundary conditions

As we have previously remarked, Donsker-Varadhan [1] have considered this case when $\rho = 1$. Therefore let us only outline how our approach proceeds for the general Dirichlet case with smooth $\rho > 0$ arbitrary to yield the conclusion (40) below. Let us assume in this section that the coefficients $a_{ij}$, $b_i$, $c$ are of class $C^\infty(\bar{\Omega})$. We shall note below where we use this assumption. We again make the change of variables $u^* = e^{-\psi}$ obtaining equations (5) - (8). However, now the boundary condition is $\psi(x) = -\ell n\, u^*(x) \to +\infty$ as $x \to \partial\Omega$. This fact causes the Dirichlet case to be somewhat more difficult and necessitates a slightly different approach than in the proof of Theorem 1.

Let $\phi$ be any positive function in $\Omega$ such that

$$\phi^2(x)/d(x) \to 0 \quad \text{as} \quad x \to \partial\Omega\,, \tag{32}$$

where $d(x)$ is the distance of $x$ to $\partial\Omega$, and $\int \rho(x)\phi^2(x)dx = 1$. For such $\phi$ let $z$ be any smooth function such that $\nabla \circ (\phi^2 z) = 0$ in $\Omega$ and let

$$w = -d + 2\phi^{-1}(\phi_x a)^T + z\,. \tag{33}$$

For such $w$, $p_w = \phi^2$ satisfies $Lp_w = 0$ with the boundary condition $p_w = 0$ on $\partial\Omega$. The operator $L$ is the same as in the proof of Theorem 1.

Multiplying (5) by $p_w$ and integrating by parts one obtains

$$(34) \qquad \int_\Omega \phi_x a\phi_x^T + c\phi^2 + \phi\phi_x b + b^T(4a)^{-1}b\phi^2 dx$$

$$+ \int_\Omega z^T(4a)^{-1}z\phi^2 + z^T(2a)^{-1}b\phi^2 + \phi\phi_x z dx \geq \lambda^* .$$

Condition (32) has been used to eliminate certain terms when integrating by parts.

For each $\phi$ we choose $z = -b + 2(W_x a)^T$ where $W$ minimizes $I(V)$ and $I(V)$ is as in the proof of Theorem 1. A simple exercise shows that

$$\text{div } (\phi^2 z) = 0$$

and

$$\int Q_x(-b + 2(W_x a)^T)\phi^2 dx = 0$$

for any smooth $Q$. Therefore we can rewrite (34) in the form

$$(35) \qquad \min_{\phi\in\Phi} \max_{V\in C^1(\bar{\Omega})} \left\{ \int_\Omega [\phi_x a\phi_x^T + c\phi^2 + \phi\phi_x b + b^T(4a)^{-1}b\phi^2]dx \right.$$

$$\left. - \frac{1}{2}\int_\Omega [(-2a)^{-1}b + V_x^T]^T(2a)[(-2a)^{-1}b + V_x^T]\phi^2 dx \right\} \geq \lambda^* ,$$

where $\Phi$ is the set of $\phi$ such that $\phi > 0$ in $\Omega$, $\phi^2/d \to 0$ as $x \to \partial\Omega$ and $\int_\Omega \rho(x)\phi^2(x)dx = 1$.

To complete our representation we wish to show that equality is obtained in (35) for some $\phi \in \Phi$. We wish to show that

$$(36) \qquad -d-b + 2(u^*)^{-1}(u_x^* a)^T = -d + 2\phi^{-1}(\phi_x a)^T + (-b + 2(W_x a)^T)$$

with $\phi \in \Phi$ and $W$ minimizes $I(V)$. As in the proof of Theorem 1 we see that $\phi$, $W$ must satisfy $\phi = u^* e^{-W}$. Since $u^*$ satisfies (1) with

$u > 0$ in $\Omega$, $u^* = 0$ on $\partial\Omega$, then $u_x^* \neq 0$ on $\partial\Omega$ and consequently there exists positive constants $c_1$, $c_2$ such that $c_1 d(x) < u^*(x) < c_2 d(x)$. Thus $(u^*)^2/d \to 0$ as $x \to \partial\Omega$. Hence if $\phi = u^* e^{-W}$, then $\phi^2/d \to 0$ as $x \to \partial\Omega$ and moreover $\phi^2 > 0$ in $\Omega$.

Now $\phi^2$, $W$ must also satisfy the equation

$$\nabla \circ (\phi^2(-b + 2(W_x a)^T) = 0 \,, \tag{37}$$

hence we must have

$$\nabla \circ ((u^*)^2 e^{-2W}(-b + 2(W_x a)^T)) = 0 \,. \tag{38}$$

Thus if the equation

$$\nabla \circ ((u^*)^2(-bG + (G_x a)^T) = 0 \tag{39}$$

has a positive solution $G$ in $\Omega$, then the change of variables $G = e^{-2W}$ defines a solution $W$ to (38). If we can find the solution $W$, then we can employ equation (36). This gives us equality in (5) and hence (35). Therefore we would have the desired result

$$\lambda^* = \min_{\phi\epsilon\Phi} \max_{V\epsilon C^1(\Omega)} \left\{ \int_\Omega \phi_x a \phi_x^T + c\phi^2 + \phi\phi_x b + b^T(4a)^{-1} b\phi^2 dx \right. \\ \left. - \frac{1}{2}\int_\Omega [(-2a)^{-1}b + V_x^T]^T (2a)[(-2a)^{-1}b + V_x^T]\phi^2 dx \right\} \,. \tag{40}$$

Thus it remains to establish the existence of a positive solution $G$ in $\Omega$ to (39). No boundary conditions are imposed with (39); this is to be expected since $(u^*)^2$ vanishes on $\partial\Omega$. Because of this last fact we employ a different approach to

establish the existence of W. For this we shall utilize the $C^\infty(\bar{\Omega})$ smoothness of $a_{ij}$, $b_i$, c although this assumption could possibly be weakened.

Consider the Ito stochastic differential equation

$$(41) \qquad d\xi = 2u^*(u^*_x a)^T + (u^*)^2(b+K)dt + \sqrt{2}\, u^*\sigma db$$

where $\sigma\sigma^T = a$, $K_i = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(a_{ij})$, with arbitrary initial data $\xi_x(0) = x \in \Omega$. We wish to show that equation (41) generates a unique ergodic measure in $\Omega$ with probability density of class $C^\infty(\Omega)$ given by G. To do this we use some results of Khasminski [3].

Consider a connected domain $D \subset \Omega$ such that $\partial D \supset \partial\Omega$ and if $x \in D$ then $0 < u^*(x) < 1/2$. Then for h sufficiently large the function $T(x) = -h\, \ell n\, u^*(x)$ satisfies in D the inequality

$$\operatorname{trace}\, (u^*)^2 a T_{xx} + T_x[(u^*)^2(b+K) + 2(u^*)(u^*_x a)^T] + 1 < 0 .$$

From standard results on Ito equations this implies that the process $\xi_x(t)$ with initial condition x in D never reaches $\partial\Omega$ and moreover hits $\partial D - \partial\Omega$ in finite mean time. Since the coefficient $\sigma$ does not vanish in $\Omega$, then results of Khasminski imply the existence of a unique finite ergodic measure. Equation (39) is the equation governing the density of the ergodic measure. Since the coefficient matrix $(u^*)^2 a$ appearing in the trace $(u^*)^2 aG_{xx}$ term in (39) is positive definite in $\Omega$, then the invariant measure has a positive density cG in $\Omega$, for an appropriate constant c, of class $C^\infty(\Omega)$. See [2], p. 248. This last fact utilizes the

smoothness assumptions on the coefficients $a_{ij}$, $b_i$, c. Hence there is a positive solution G and the outline of the derivation of (40) is concluded.

For the Donsker-Varadhan analogue of (40) in the Dirichlet case with $\rho = 1$, see equation (3.7) in [1] and note that a/2 instead of a is used in their version of (1).

## 4. Numerical methods

Let us propose a numerical method for computing approximations to the smallest eigenvalue $\lambda^*$ and the corresponding eigenfunction $u^*$ for the Neumann problem and also for the periodic problem which was not considered in Sections 1-3. For details in a simple case, see Holland [6]. The method consists in applying a finite difference approximation to the nonlinear equation (5) and the corresponding boundary condition. If $\rho = 1$, the resulting nonlinear difference equations can be given an interpretation as the equations for optimal stationary control of a Markov chain.

We have employed a method due to White for solution of the Markov chain problem. White's method was suggested by Kushner as a possible method, see Kushner [5], pp. 156-157 for a description of the method. Preliminary results show that this approach is satisfactory for obtaining $\lambda^*$ and $u^*$ for the periodic and Neumann problems. Our present numerical work includes periodic problems in 2 spatial dimensions and Neumann problems in 1 spatial dimension. Some preliminary numerical work on the Dirichlet problem also indicates that either the formal approximation suggested in [6] is invalid or the rate of convergence is so slow that the method is not useful.

We have not yet justified the validity of this discretization process. Kushner has justified this approach for other problems, see [4] for a summary.

Acknowledgments

The author would like to express his appreciation to M. Donsker, H. McKean, Jr., L. Nirenberg of the Courant Institute and S. Baouendi of Purdue University for some helpful remarks concerning Sections 1-3. Finally I wish to thank H. Kushner of Brown University for some helpful remarks concerning Section 4.

Footnote to page 1:

* This research was supported in part by a grant to the Courant Institute of Mathematical Sciences from the Alfred P. Sloan Foundation and in part by AFOSR Grant 77-3286 to Purdue University.

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